Indian Statistical Institute, Bangalore Centre
B.Math(Hons.) II Year, First Semester
Mid-Sem Examination
Analysis III

Time: 3 Hours September 23, 2011 Instructor: Pl.Muthuramalingam

Total Mark is : 40

Note: The paper has part A and B. Maximum marks you can get in part B is 30.

Part A

1. Let $P(x,y) = xy^2$, $Q(x,y) = 2x^2y$

   (a) Find $\int\limits_{\Gamma_j} P\ dx + Q\ dy$ for $j = 1, 2, 3$ where

   $$\begin{aligned} \Gamma_1 &= \text{the line segment joining } (0,0) \text{ to } (3,0) \\ \Gamma_2 &= \text{the line segment joining } (0,2) \text{ to } (0,0) \\ \Gamma_3 &= \text{the curve } \{4x^2 + 9y^2 = 36, x > 0, y > 0\} \end{aligned}$$

   [1+1+2]

   (b) Let $G = \{4x^2 + 9y^2 \leq 36,\ x > 0,\ y > 0\}$. Find $\int\limits_G\int \frac{\partial Q}{\partial x}\ dxdy$ and $\int\limits_G\int \frac{\partial P}{\partial y}\ dxdy$ [2+2]

   (c) Verify Greens theorem for $P,\ Q,\ G,\ \partial G$. [2]

Part B

2. (a) Let G be an oval in R^2 given by

   $$\begin{aligned} G &= \{(x,y)\ :\ a \leq x \leq b,\quad f_1(x) \leq y \leq f_2(x)\} \\ &= \{(x,y)\ :\ c \leq y \leq d,\quad g_1(y) \leq x \leq g_2(y)\} \end{aligned}$$

   Here $f_1, f_2 : [a,b] \to R$ are C^1 functions with $f_1(a) = f_2(a), f_1(b) = f_2(b)$. Also $g_1, g_2 : [c,d] \to R$ are C^1 functions with $g_1(c) = g_2(c), g_1(d) = g_2(d)$. State and prove Greens theorem for G and boundary of G described in a suitable manner. [5]

   (b) For any curve $\Gamma : [a,b] \to R^2$, define opposite of $\Gamma = \Gamma_1$ by $\Gamma_1(t) = \Gamma(b - t + a)$ for t in $[a,b]$. For continuous functions $P, Q : R^2 \to R$ find a relation between $\int\limits_\Gamma Pdx + Qdy$ and $\int\limits_{\Gamma_1} Pdx + Qdy$ and prove your claim. [2]

3. Let (X, d) be a metric space. $f_1, f_2, \ldots : (X, d) \to R$ are all continuous and bounded. Let $g : X \to R$ be any bounded function such that

$$0 = \lim_{k\to\infty} \sup_{x\in X} |f_k(x) - g(x)|$$

Show that g is a continuous function. [5]

4. Let $f_1, f_2, \ldots$ be a sequence of bounded Riemann integrable functions on $[a, b]$. Let $g : [a, b] \to R$ be a bounded function. Assume that

$$0 = \lim_{k\to\infty} \sup_{t\in[a,b]} |f_k(t) - g(t)|.$$

Show that g is Riemann integrable on $[a, b]$ [5]

5. Let $f_n(t) = n^2\, t(1-t)^n$ for t in $[0, 1], n = 1, 2, 3, \ldots$

   (a) Show that $\lim_{n\to\infty} f_n(t) = 0$ [2]

   (b) Show that $\lim_{n\to\infty} \int_0^1 f_n(t)dt \neq 0$ [3]

6. Let $\mathcal{P} = \{a = t_0 < t_1 < t_2 < \ldots < t_n = b\}$ be a partition of $[a, b]$. A is a set adapted to $\mathcal{P}$ if

$$A = \{s_0, s_1, s_2, \ldots, s_{n-1}\}$$

where $s_j \in [t_j, t_{j+1}]$. For $f : [a, b] \to R$ any bounded function and $\alpha : [a, b] \to R$ any increasing function, define $S(\mathcal{P}, A, f, \alpha)$ by

$$S(\mathcal{P}, A, f, \alpha) = \sum f(s_i)[\alpha(t_{i+1}) - \alpha(t_i)].$$

Let $\alpha_0(t) = t$. Assume that α is differentiable and α' is continuous. Let $g(t) = f(t)\ \alpha'(t)$ on $[a, b]$. Show that for any partition $\mathcal{P}$ there exists a set A such that

$$S(\mathcal{P}, A, f, \alpha) = S(\mathcal{P}, A, g, \alpha_0)$$

[2]

7. Let $a < x_0 < b$. Define an increasing function $\alpha : [a, b] \to R$ by $\alpha(t) = p_1$ on $[a, x_0)$, $\alpha(x_0) = p_2$, $\alpha(t) = p_3$ on $(x_0, b]$ where $p_1 < p_2 < p_3$. Let $f : [a, b] \to R$ be any bounded function.

   (a) If $f \in \mathcal{RS}(\alpha)$, then f is continuous at x_0. [3]

   (b) If f is continuous at x_0, then $f \in \mathcal{RS}(\alpha)$. [2]

   (c) Let (b) hold. Then $\int_a^b f d\alpha = f(x_0)(p_3 - p_1)$. [2]

8. Let $f : [a,b] \times [c,d] \to R$ be any continuous function. Define $g : [a,b] \to R$ by

$$g(x) = \int_c^d f(x,y)\ dy.$$

(a) Show that g is uniformly continuous. [2]

(b) Show that $\int_a^b g(x)\ dx$ can be approximated by a double sum of the form

$$\sum f(u_i, v_j)\ (t_{i+1} - t_i)\ (s_{j+1} - s_j)$$

where
$a = t_0 < t_1 < \ldots\ldots < t_n = b,$
$c = s_0 < s_1 < \ldots\ldots < s_p = d,$
$t_i \leq u_i \leq t_{i+1}$ and $s_j \leq v_j \leq s_{j+1}.$ [5]

9. Let $f_n(t) = \frac{t^{2n}}{1+t^{2n}}$ for t in R, $n = 1, 2, 3, \ldots$.

Show that $f(t) = \lim_{n\to\infty} f_n(t)$ exists. [1]

Show that f is not continuous. [1]